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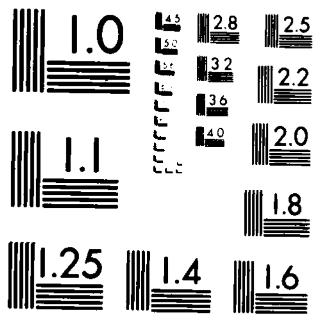

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PERIODIC SOLUTIONS OF LAGRANGIAN SYSTEMS ON A COMPACT MANIFOLD

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ABSTRACT

Let M be a smooth n-dimensional manifold and let TM be its tangent bundle. We consider a time periodic Lagrangian of period T,

$$L_t : TM \to R$$

and we seek T-periodic solutions of the Lagrange equations, which in local coordinates are

$$(*) \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(t,q,\dot{q}) - \frac{\partial L}{\partial q}(t,q,\dot{q}) = 0 \qquad i = 1,\dots,n .$$

Our main result states that if the fundamental group of M is finite, then (*) has infinitely many T-periodic solutions, provided that L_t satisfies certain physically reasonable assumptions.

AMS (MOS) Subject Classifications: 58E05, 58F05, 70H35, 34C25

Key Words: Lagrangian system, tangent bundle, infinite dimensional manifold, critical point, cohomology algebra, assumption c of Palais and Smale

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

The question of existence and the number of periodic solutions of model equations for a classical mechanical system is a problem as old as the field of analytical mechanics itself. The development of the nonlinear functional analysis has renewed interest in these problems.

In this paper we consider a mechanical system which is constrained to a compact manifold M. We suppose that the dynamics of the system is described by a T-periodic Lagrangian

L subt ; TM approaches R

$$L_t : TM \to R$$

which satisfies reasonable physical assumptions. The main result of this paper is: If the fundamental group of the manifold M is finite, then the Lagrangian nonlinear system of differential equations which describes the dynamical system has infinitely many distinct periodic solutions.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

PERIODIC SOLUTIONS OF LAGRANGIAN SYSTEMS ON A COMPACT MANIFOLD

Vieri Benci

INTRODUCTION

The existence and the number of periodic solutions of model equations for a classical mechanical system is a problem as old as the field of analytical mechanics itself. The development of the nonlinear functional analysis has renewed interest in these problems (we refer to [R] for a recent bibliography on the subject).

In this paper we are interested in periodic solutions of prescribed period when the system is constrained to a compact manifold. This fact allows us to use many tools developed in the theory of closed geodesics on Riemannian compact manifolds (cf. [K]). We now describe our results.

Let M be a smooth n-dimensional manifold and let TM be its tangent bundle. We consider a time-dependent Lagrangian

$$L_t : TM \to \mathbb{R}$$

We suppose that L_t is T-periodic in time and we seek T-periodic solution $\gamma(t) \in M$ of the corresponding dynamical system. We fix a finite C^∞-atlas

(0.1)(a) $$A = \{U_\ell, \phi_\ell\}_{\ell=1,\ldots,N} \quad \text{for } M$$

and the corresponding atlas

(0.1)(b) $$TA = \{TU_\ell, T\phi_\ell\}_{\ell=1,\ldots,N} \quad \text{for } TM$$

So in local coordinates, our dynamical system is described by the following system of second order differential equations:

(0.2) $$\frac{d}{dt}\frac{\partial L_\ell}{\partial v_i}(t,q(t),\dot q(t)) - \frac{\partial L_\ell}{\partial q_i}(t,q(t),\dot q(r)) = 0$$

$$\text{for } i = 1,\ldots,n \text{ and } \gamma(t) \in U_\ell, \quad \ell = 1,\ldots,N$$

where

(0.3) $$L_\ell(t,q,v) = L_t \circ (T\phi_\ell)^{-1}(q,v) \quad \text{and} \quad (q(t),\dot q(t)) = (T\phi_\ell)\dot\gamma .$$

We shall suppose that $T = 1$ (if not it is sufficient to rescale the time) and we set

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$S^1 = \mathbb{R}/\mathbb{Z}$ so that we can regard a solution of (0.1) as a function $\gamma : S^1 \to M$. We make the following assumption on L

(L_0) L_ℓ is twice differentiable for $\ell = 1,\dots,N$

There exists a constant $c > 0$ such that

$$(L_1) \quad (a) \quad \left|\frac{\partial L_\ell}{\partial q_i}(t,q,v)\right| \le c(1 + |v|^2)$$

$$(b) \quad \left|\frac{\partial L_\ell}{\partial v_i}(t,q,v)\right| \le c(1 + |v|)$$

$$(L_2) \quad (a) \quad \left|\frac{\partial^2}{\partial q_i \partial q_j} L_\ell(t,q,v)\right| \le c(1 + |v|^2)$$

$$(b) \quad \left|\frac{\partial^2}{\partial q_j \partial v_j} L_\ell(t,q,v)\right| \le c(1 + |v|)$$

$$(c) \quad \left|\frac{\partial^2}{\partial v_i \partial v_j} L_\ell(t,q,v)\right| \le c$$

for $i,j = 1,\dots,n$ and $\ell = 1,\dots,N$.

(L_3) there exists a constant $\nu > 0$ such that

$$\sum_{ij} \frac{\partial^2}{\partial v_i \partial v_j} L(t,q,v) w_i w_j \ge \nu |w|^2 \quad \text{for} \quad \ell = 1,\dots,N$$

For example the Lagrangian defined by

$$L_\ell(t,q,v) = \sum_{ij} a^\ell_{ij}(t,q) v_i v_j + \sum_i b^\ell_i(t,q) v_i + c^\ell(t,q)$$

satisfies (L_1), (L_2) and (L_3) if $a^\ell_{i,j}$, b^ℓ_i, $c^\ell \in C^2(U_\ell)$ and the matrix $\{a^\ell_{ij}(t,q)\}$ is positive infinite for every $t \in S^1$ and $q \in U_\ell$.

We say that a periodic solution of (0.2) is homotopically trivial (resp. nontrivial), if the map $\gamma : S^1 \to M$ is homotopically trivial (resp. nontrivial).

The main result of this paper is the following one

0.1 Theorem. *Suppose that* L_t *satisfies* (L_1) (L_2) (L_3), *then*

(i) *for each conjugacy class of the fundamental group of* M *there exists at least a homotopically nontrivial periodic solution of (0.2)*

(ii) *if the fundamental group of* M *is finite, then there exist infinitely many homotopically trivial periodic solutions of (0.2).*

The result of Theorem (0.1) is optimal as the following example shows. Take $M = S^1 = \mathbb{R}/\mathbb{Z}$; $L_t = \langle\cdot,\cdot\rangle$ where $\langle\cdot,\cdot\rangle$ is the standard Riemanian structure on S^1. Then all the 1-periodic solutions of (0.2) have the form $\gamma(t) = rt$ $(r \in \mathbb{Z})$. Since $\pi_1(S^1) = \mathbb{Z}$, this simple example shows that

(i) to each conjugacy class of $\pi_1(M)$ may correspond only one periodic solution of (0.2).

(ii) if $\pi_1(M)$ is infinite we may not have any homotopically trivial periodic solution of (0.2).

By Theorem 0.1 the following corollary follows

0.2 Corollary. *If* M *is a Lie group (or more in general a H-space) then 0.2 has infinitely many periodic solutions.*

Proof. Under our assumptions $\pi_1(M)$ is an Abelian group. Then if it is infinite, the conclusion follows by Theorem 0.1 (i); if it is finite, the conclusion follows from Theorem 0.1 (ii).

We thank E. Fadell and J. Robbin for many useful conversations on this topic.

1. DESCRIPTION OF THE FUNCTION SPACES USED

Let M be a smooth compact manifold of dimension N and let $S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/\{0,1\}$. For $s \in (1/2,+\infty]$ we set

$$\Lambda^s M = W^s(S^1,M)$$

where $W^s(S^1,M)$ denotes the Sobolev space of functions $\gamma : S^1 \to M$ of order s. Since there exists n' such that $M \subset \mathbb{R}^{n'}$, the easiest way to define $W^s(S^1,M)$ probably is the following one:

$$W^s(S^1,M) = \{\gamma \in W^s(S^1,\mathbb{R}^{n'}) | \gamma(t) \in M \text{ for every } t\}$$

We remark that the above assumption makes sense. In fact since $s > 1/2$, by the Sobolev embedding theorem, the function in $W^s(S^1,M)$ are continuous. If $s < 1/2$ there is not any reasonable definition (cf. e.g. [A]).

$W^s(S^1,M)$ can also be defined using the atlas (0.1)(a). We say that $\gamma \in W^s(S^1,M)$ if for every interval $\tau \subset S^1$ such that $\gamma(\tau) \in U_i$, we have that

$$\phi_i \circ \gamma|_\tau : \tau \to R^k \text{ is a function in } W^s(S^1,\mathbb{R}^k); \quad (U_i,\phi_i) \in A$$

Palais has shown that the two definitions are equivalent [Pa]. We will be interested in the two cases when $s = 1$ or $s = +\infty$. In these cases we set

$$\Lambda^1 M = W^1(S^1,M) = \text{function with "square integrable derivative"}$$

and

$$\Lambda^\infty M = W^\infty(S^1,M) = C^\infty(S^1,M) = \text{functions continuous with all their derivatives.}$$

It is well known that $\Lambda^1 M$ is a Hilbert manifold (cf. e.g. [Pa], [K], [A]). We also need to use the space $C(S^1,M)$ of the continuous functions $\gamma : S^1 \to M$. We shall use the following notation

$$\Lambda M = C(S^1,M)$$

It is well known that ΛM is a Banach manifold (cf. e.g. [K]). Now consider the tangent bundle $TM \xrightarrow{\pi} M$. For $s \in (1/2,\infty)$ and $r \leq s$ define

$$T^r\Lambda^s M = \{\xi : S^1 \to TM : \xi \text{ is a vector field of class } W^r \text{ along a curve } \gamma \in \Lambda^s M\}$$

If we define a map $\tilde{\pi} : T^r\Lambda^s M \to \Lambda^s M$ as follows

$$(\tilde{\pi}\xi)(t) = \pi(\xi(r)) \text{ for a.e. } t \in S^1$$

it follows that $\{T^r\Lambda^s M, \tilde{\pi}, \Lambda^s M\}$ is a (infinite dimensional) vector bundle over $\Lambda^s M$ (cf. [K] or [A] for proofs and details). In particular, for $r = s$, we obtain the tangent bundle of $\Lambda^s M$. In this case we shall write simple $T\Lambda^s M$. Also we shall use the following notation

$$T^r_\gamma \Lambda^s M = (\tilde{\pi})^{-1}\gamma = \{\xi : S^1 \to M \mid \xi \text{ is a vector of class } W^r \text{ along } \gamma\}$$

$$T_\gamma \Lambda^s M = (\tilde{\pi})^{-1}\gamma = \{\xi : S^1 \to M \mid \xi \text{ is a vector of class } W^s \text{ along } \gamma\}$$

Similarly we define

$$T\Lambda M = \{\xi : S \to M \mid \xi \text{ is a continuous vector field along a curve } \gamma \in \Lambda M\}$$

$$T^{\#}\Lambda^s M = \{\xi : S \to M \mid \xi \text{ is a continuous vector field along a curve } \gamma \in \Lambda^s M\} \quad s > 1/2$$

By well known theorems on Sobolev spaces, we have that the embeddings.

$T_\gamma\Lambda^1 M \longrightarrow T^{\#}_\gamma\Lambda^1 M \longrightarrow T^0_\gamma\Lambda^1 M$ are continuous and the first one is also compact (for detail see e.g. [K] or [A]). In order to make easier the computation in the following sections it is useful to introduce a Riemann structure $\langle\ ,\ \rangle$ on M. This structure permits to define Hilbert structures on $T^0_\gamma\Lambda^1 M$ and $T^1_\gamma\Lambda^1 M$ as follows

$$\langle\xi,\eta\rangle_0 = \int_0^1 \langle\xi(t),\eta(t)\rangle_{\gamma(t)}dt \qquad \xi,\eta \in T^0_\gamma\Lambda^1 M$$

$$\langle\xi,\eta\rangle_1 = \int_0^1 \{\langle\nabla_t\xi(t),\nabla_t\eta(t)\rangle_{\gamma(t)} + \langle\xi(t),\eta(t)\rangle_{\gamma(t)}\}dt \qquad \xi,\eta \in T_\gamma\Lambda^1 M$$

where ∇_t denotes the covariant derivative. We shall use also the following notation

$$\|\xi\|_0 = \langle\xi,\xi\rangle_0^{1/2} (\xi \in T^0_\gamma\Lambda^1 M) \quad \text{and} \quad \|\xi\|_1 = \langle\xi,\xi\rangle_1^{1/2} (\xi \in T_\gamma\Lambda^1 M)$$

We also define

$$\|\xi\|_{\#} = \Big[\sup_{t\in(0,1)} \langle\xi(t),\xi(t)\rangle_{\gamma(t)}\Big]^{1/2} \quad \text{for} \quad \xi \in T_\gamma\Lambda M$$

The above definition allow to define the following distances on $\Lambda^1 M$

$$\mathrm{dist}_1(\gamma_1,\gamma_2) = \min_{\beta\in\mathcal{B}} \int_0^1 \|\dot\beta(\lambda)\|_1 d\lambda$$

$$\mathrm{dist}_0(\gamma_1,\gamma_2) = \min_{\beta\in\mathcal{B}} \int_0^1 \|\dot\beta(\lambda)\|_0 d\lambda$$

where $\mathcal{B}$ is the set of curves $\beta(\lambda)$ of class C^1 joining γ_1 and γ_2 and

$$\dot{\beta}(\lambda) = \frac{d}{d\lambda}\beta .$$

It turns out that $\Lambda^1 M$ is a complete metric space with respect to the distance $d_1(\cdot,\cdot)$. Actually it is an infinite dimensional Riemann manifold with respect to the Riemann structure $\langle\cdot,\cdot\rangle_1$ and the topology induced by this metric is the same given by the definition (cf. [K] for proofs and details).

We also define for $\gamma_1, \gamma_2 \in \Lambda M$

$$\mathrm{dist}_*(\gamma_1,\gamma_2) = \min_{\beta\in\mathcal{B}} \int_0^1 \|\dot{\beta}(\lambda)\|_* d\lambda$$

where $\mathcal{B} = \{\beta \in C^1([0,1],\Lambda M) : \beta(0) = \gamma_1, \beta(1) = \gamma_2\}$.

As expected it turns out that ΛM is a complete metric space with the distance $\mathrm{dist}_*(\gamma_1,\gamma_2)$ and the topology given by this metric is the uniform convergence topology.

By virtue of the compactness of the embedding $\Lambda^1 M \longrightarrow \Lambda M$ the following result holds (see [K] for details).

Lemma 1.1. If $\{\gamma_n\}$ is a sequence in $\Lambda^1 M$, bounded with respect to the metric $d_1(\cdot,\cdot)$, then it has a subsequence converging in ΛM.

2. ESTIMATES OF THE ACTION FUNCTIONAL ON $\Lambda^1 M$

At least formally, the solutions on (0.2) are the critical point of the action functional

(2.1) $$f(\gamma) = \int_{S^1} L_t \circ \gamma \, dt$$

We shall show that the functional (2.1) is a functional of class C^2 on $\Lambda^1 M$. In this section we shall prove this fact and we shall give some estimates to be used later.

In order to carry out this program it is useful to have nice local representations of the quantity involved by means of the atlas (0.1)(a), (b). In this way it will be possible to exploit assumptions (L_1), (L_2) and (L_3). For $\gamma \in \Lambda^1 M$, we divide S^1 in "intervals" $\tau_1, \ldots, \tau_p$ (where p depends on γ) such that

$$\gamma(t) \in U_\ell \quad \text{for} \quad t \in \tau_\ell \qquad \ell = 1, \ldots, p$$

where $\{U_\ell, \phi_\ell\}$ is a chart of the atlas (0.1)(a). Then we set

(2.2) $$q_\ell = \phi_\ell \circ \gamma|_{\tau_\ell} \qquad \ell = 1, \ldots, p .$$

Clearly $q_\ell \in W^1(\tau_\ell, R^n)$ and

(2.3) $$\|q_\ell\|_{L^\infty(\tau_\ell, R^n)} \leq c_1$$

where c_1 is a constant which depends only on the atlas (0.1)(a). Moreover

$$\dot{\gamma}(t) \in TU_\ell \quad \text{for} \quad t \in \tau_\ell \quad \text{and} \quad \ell = 1, \ldots, p ;$$

then we set

$$(q_\ell, \dot{q}_\ell) = T\phi_\ell \circ \dot{\gamma}|_{\tau_\ell} \qquad \ell = 1, \ldots, p .$$

Clearly we have

$$\dot{q}_\ell = \frac{d}{dt} q_\ell$$

and

(2.5) $$\dot{q}_\ell \in L^2(\tau_\ell, R^n) .$$

If $\xi \in T_\gamma \Lambda^1 M$, we have that

$$\xi(t) \in TU_\ell \quad \text{for} \quad t \in \tau_\ell \quad \text{and} \quad \ell = 1, \ldots, p ;$$

then we set

(2.5') $$(q_\ell, \delta q_\ell) = T\phi_\ell \circ \xi|_{\tau_\ell} \quad \text{for} \quad \ell = 1, \ldots, p .$$

By the definition of $T_\gamma\Lambda^1M$ we have that

$$\delta q_\ell \in W^1(\tau_\ell,\mathbb{R}^n) \ . \tag{2.6}$$

Moreover $\dot\xi \in T^2U_\ell$ where T^2 denotes the "double tangent" operator. So we can set

$$(q_\ell,\ \delta q_\ell,\ \dot q_\ell,\ \delta\dot q_\ell) = T^2\phi_\ell \circ \dot\xi|_{\tau_\ell} \tag{2.7}$$

Of course q_ℓ and $\dot q_\ell$ defined by the above formula agree with q_ℓ and $\dot q_\ell$ given by (2.4) and

$$\delta\dot q_\ell = \frac{d}{dt}\,\delta q \in L^2(\tau_\ell,\mathbb{R}^n) \ . \tag{2.8}$$

<u>Definition 2.1</u>. <u>Given</u> $\gamma \in \Lambda^1M$ and $\xi \in T_\gamma\Lambda^1M$, <u>we shall call the functions</u> $q_\ell,\ \dot q_\ell,\ \delta q_\ell,\ \delta\dot q_\ell$ <u>(defined by (2.2), (2.4), (2.5') and (2.7)) a A-local representation of</u> $\gamma,\ \dot\gamma,\ \xi$ and $\dot\xi$ <u>respectively. Also we shall call the corresponding functions</u> $L_\ell(t,q,v)$ <u>for</u> $\ell = 1,\dots,p$ <u>(given by (0.3)) a A-local representation of</u> $\mathbb{L}_t$ <u>corresponding to</u> γ.

Using a A-local representation of $\dot\gamma$ and of $\mathbb{L}_t$ the functional (2.1) takes the following form

$$f(\gamma) = \sum_{k=1}^{p}\int_{\tau_\ell} L_\ell(t,q_\ell(t),\dot q_\ell(t))\,dt \tag{2.9}$$

<u>Lemma 2.2</u> <u>Let</u> L_ℓ <u>be a function given by (0.3) and set</u>

$$g_\ell(q) = \int_{\tau_\ell} L_\ell(t,q,\dot q)\,dt \tag{2.10}$$

<u>where</u> τ_ℓ <u>is a subinterval of [0,1]. If</u> L_ℓ <u>satisfies</u> (L_1), (L_2) <u>then</u> g_ℓ <u>is a functional of class</u> C^2 on $W^1(\tau_\ell,\mathbb{R}^n)$ <u>with</u>

$$g_\ell'(q)[\delta q] = \sum_i \int_{\tau_\ell} \left\{\frac{\partial L_\ell}{\partial q_i}(t,q,\dot q)\,\delta q + \frac{\partial L}{\partial v}(t,q,\dot q)\,\delta\dot q\right\}dt \tag{2.11}$$

$$g_\ell''(q)[\delta q]^2 = \sum_{ij}\int_{\tau_\ell}\left\{\frac{\partial^2}{\partial q_i\partial q_j}L_\ell(t,q,\dot q)\,\delta q_i\,\delta q_j + 2\,\frac{\partial^2}{\partial q_i\partial v_j}L(t,q,\dot q)\,\delta q_i\,\delta\dot q_j\right. \tag{2.12}$$

$$\left. + \frac{\partial^2}{\partial v_i\partial v_j}L_\ell(t,q,\dot q)\,\delta\dot q_i\,\delta\dot q_j\right\}dt$$

and the following inequalities are satisfied

$$g_\ell'(q)[\delta q]^2 \le c_1(|\tau_\ell| + \|q\|^2_{W^1(\tau_\ell,R^n)})\|\delta q\|_{W^1(\tau_\ell,R^n)} \tag{2.13}$$

$$g_\ell''(q)[\delta q]^2 \le c_2(|\tau_\ell| + \|q\|^2_{W^1(\tau_\ell,R^n)})\|\delta q\|_{W^1(\tau_\ell,R^n)} \tag{2.14}$$

where c_1 *and* c_2 *depend only on the constant* c *appearing in* (L_2) *and* $|\tau_\ell|$ *is the measure of* τ_ℓ. *Moreover if* (L_3) *holds we have*

$$g_\ell''(q)[\delta q]^2 \ge a\|\delta q_\ell\|^2_{W^1(\tau_\ell,R^n)} - b(|\tau_\ell| + \|q_\ell\|^2_{W^1(\tau_\ell,R^n)})\|\delta q_\ell\|^2_{L^\infty(\tau_\ell,R^n)} \tag{2.15}$$

where the constants a *and* b *depend only on the constants* c *and* ν *appearing in* L_2 *and* L_3.

Proof. Clearly, (2.11) and (2.12) hold formally. Therefore we just have to prove inequalities (2.13) and (2.14). In the following $c_3, c_4, \ldots$ will denote suitable positive constants. By $(L_1)(a)$ we have

$$\left|\int_{\tau_\ell} \frac{\partial L_\ell}{\partial q_i}\delta q_i dt\right| \le c\int_{\tau_\ell}(1 + |\dot q|^2)|\delta q|dt \le \tag{2.16}$$

$$\le c(|\tau_\ell| + \|q\|^2_{W^1})\|\delta q\|_{L^\infty}$$

$$\le c_3(|\tau_\ell| + \|q\|^2_{W^1})\|\delta q\|_{W^1}$$

By $(L_1)(b)$ we have

$$\left|\int_{\tau_\ell} \frac{\partial L_\ell}{\partial v_i}\delta\dot q_i dt\right| \le c\int_{\tau_\ell}(1 + |\dot q|)|\delta\dot q|dt \tag{2.17}$$

$$\le c(|\tau_\ell| + \|q\|_{W^1})\left(\int_{\tau_\ell}|\delta q|^2\right)^{1/2} \qquad \text{(by Schwartz inequality)}$$

$$\le c_4(|\tau_\ell| + \|q\|^2_{W^1})\|\delta q\|_{W^1}$$

By (2.16) and (2.17), (2.13) follows.

By (L_2)(a) we have

$$(2.18)\qquad \left|\int_{\tau_\ell} \frac{\partial^2 L}{\partial q_i \partial q_j}\, \delta q_i \delta q_j dt\right| \le c \int_{\tau_\ell} (1 + |\dot q|^2)|\delta q|^2 dt$$

$$\le c(|\tau_\ell| + \|q\|^2_{W^1})\|\delta q\|^2_{L^\infty}$$

By (L_2)(b) we have

$$(2.19)\qquad 2\left|\int_{\tau_\ell} \frac{\partial^2 L}{\partial q_i v_j}\, \delta q_i \delta \dot q_j dt\right| \le 2c \int_{\tau_\ell} (1 + |\dot q|)|\delta q||\delta \dot q| dt$$

$$\le 2c\|\delta q\|_{L^\infty}\Big(\int_{\tau_\ell} (1 + |\dot q|)^2 dt\Big)^{1/2}\cdot\Big(\int_{\tau_\ell} |\delta \dot q|^2 dt\Big)^{1/2} \quad \text{(by Schwartz inequality)}$$

$$\le c_4(|\tau_\ell| + \|q\|_{W^1})\cdot\|\delta q\|_{L^\infty}\cdot\|\delta q\|_{W^1}$$

By (L_2)(c) we have

$$\left|\int_{\tau_\ell} \frac{\partial^2 L}{\partial v_i \partial v_j}\, \delta \dot q_i \delta \dot q_j\right| \le c\|\delta \dot q\|^2_{W^1}$$

By the above inequality, (2.18) and (2.19), (2.14) follows.

By (L_3) we have

$$\sum_{ij} \int_{\tau_\ell} \frac{\partial^2 L}{\partial v_i v_j}\, \delta \dot q_i \delta \dot q_j dt \ge \nu \int_{\tau_\ell} |\delta \dot q|^2 =$$

$$= \nu\|\delta q\|^2_{W^1} - \|\delta q\|^2_{L^2} \ge \nu\|\delta q\|^2_{W^1} - \|\delta q\|^2_{L^\infty}$$

By the above inequality, (2.19) and (2.18) we have

(2.20) $$g_\ell''(q)[\delta q]^2 \geq \nu\|\delta q\|^2_{W^1} - \|\delta q\|^2_{L^\infty} - c_4(|\tau_\ell| + \|q\|_{W^1})\|\delta q\|_{L^\infty}\|\delta q\|_{W^1}$$

$$- c(|\tau_\ell| + \|q\|^2_{W^1})\|\delta q\|^2_{L^\infty}$$

Since

$$c_4(|\tau_\ell| + \|q\|_{W^1})\|\delta q\|_{L^\infty}\|\delta q\|_{W^1} \leq \frac{\nu}{2}\|\delta q\|^2_{W^1} + \frac{-}{2\nu}c_4^2(|\tau_\ell| + \|q\|_{W^1})^2\|\delta q\|^2_{L^\infty}$$

$$\leq \frac{\nu}{2}\|\delta q\|^2_{W^1} + c_5(|\tau_\ell| + \|q\|^2_{W^1})\|\delta q\|^2_{L^\infty}$$

(We have used the fact that $|\tau_\ell|^2 \leq |\tau_\ell| \leq 1$). By the above inequality and (2.20) we get

$$g_\ell''(q)[\delta q]^2 \geq \frac{\nu}{2}\|\delta q\|^2_{W^1} - (1 + c_5 + c)(|\tau_\ell| + \|q\|^2_{W^1})\|\delta q\|^2_{L^\infty}$$

By the above inequality, (2.14) follows. □

<u>Lemma 2.3</u>. <u>The functional</u> f <u>defined by (2.1) is a</u> C^2-<u>functional on</u> $\Lambda^1 M$. <u>Moreover if</u> q_ℓ, δq_ℓ <u>is a local</u> Λ-<u>representation of</u> γ <u>and</u> ξ <u>we have</u>

(2.21) $$f'(\gamma)[\xi] = \sum_\ell g_\ell'(q_\ell)[\delta q_\ell]$$

(2.22) $$f''(\gamma)[\xi]^2 = \sum_\ell g_\ell''(q_\ell)[\delta q_\ell]^2$$

<u>where</u> g_ℓ <u>is defined in lemma 2.2</u>

<u>Proof</u>. Let $\beta(\lambda)$ $(\lambda \in (u - \varepsilon, u + \varepsilon), \varepsilon > 0)$ be a C^1-curve in $\Lambda^1 M$ such that $\beta(0) = \gamma$, $\frac{d}{d\lambda}\beta(\lambda) = \xi$ and let q_ℓ, δq_ℓ be a Λ-local representation of γ and ξ.

Then, using (2.9) and lemma 2.2 we get

$$\frac{d}{d\lambda} f(\beta(\lambda))\Big|_{\lambda=0} = \sum_\ell g_\ell'(q_\ell)[\delta q_\ell]$$

$$\frac{d^2}{d\lambda^2} f(\beta(\lambda))\Big|_{\lambda=0} = \sum_\ell g_\ell''(q_\ell)[\delta q_\ell]^2$$

The above formulas prove (2.21) and (2.22). □

In carring out our estimates on the functional f it is useful to make use of the Riemann structure $\langle , \rangle$ on M which, as we have seen in Section 1, induces a infinite dimensional Riemann structure $\langle \cdot , \cdot \rangle_1$ on $\Lambda^1 M$.

$$(2.20) \qquad g''_\ell(q)[\delta q]^2 \geq \nu\|\delta q\|^2_{W^1} - \|\delta q\|^2_{L^\infty} - c_4(|\tau_\ell| + \|q\|_{W^1})\|\delta q\|_{L^\infty}\|\delta q\|_{W^1}$$

$$- c(|\tau_\ell| + \|q\|^2_{W^1})\|\delta q\|^2_{L^\infty}$$

Since

$$c_4(|\tau_\ell| + \|q\|_{W^1})\|\delta q\|_{L^\infty}\|\delta q\|_{W^1} \leq \frac{\nu}{2}\|\delta q\|^2_{W^1} + \frac{1}{2\nu}c_4^2(|\tau_\ell| + \|q\|_{W^1})^2\|\delta q\|^2_{L^\infty}$$

$$\leq \frac{\nu}{2}\|\delta q\|^2_{W^1} + c_5(|\tau_\ell| + \|q\|^2_{W^1})\|\delta q\|^2_{L^\infty}$$

(We have used the fact that $|\tau_\ell|^2 \leq |\tau_\ell| \leq 1$). By the above inequality and (2.20) we get

$$g''_\ell(q)[\delta q]^2 \geq \frac{\nu}{2}\|\delta q\|^2_{W^1} - (1 + c_5 + c)(|\tau_\ell| + \|q\|^2_{W^1})\|\delta q\|^2_{L^\infty}$$

By the above inequality, (2.14) follows. □

<u>Lemma 2.3</u>. <u>The functional</u> f <u>defined by (2.1) is a</u> C^2-<u>functional on</u> $\Lambda^1 M$. <u>Moreover if</u> q_ℓ, δq_ℓ <u>is a local A-representation of</u> γ <u>and</u> ξ <u>we have</u>

$$(2.21) \qquad f'(\gamma)[\xi] = \sum_\ell g'_\ell(q_\ell)[\delta q_\ell]$$

$$(2.22) \qquad f''(\gamma)[\xi]^2 = \sum_\ell g''_\ell(q_\ell)[\delta q_\ell]^2$$

<u>where</u> g_ℓ <u>is defined in lemma 2.2</u>

<u>Proof</u>. Let $\beta(\lambda)$ $(\lambda \in (u - \varepsilon, u + \varepsilon), \varepsilon > 0)$ be a C^1-curve in $\Lambda^1 M$ such that $\beta(0) = \gamma$, $\frac{d}{d\lambda}\beta(\lambda) = \xi$ and let q_ℓ, δq_ℓ be a A-local representation of γ and ξ.

Then, using (2.9) and lemma 2.2 we get

$$\frac{d}{d\lambda} f(\beta(\lambda))\Big|_{\lambda=0} = \sum_\ell g'_\ell(q_\ell)[\delta q_\ell]$$

$$\frac{d^2}{d\lambda^2} f(\beta(\lambda))\Big|_{\lambda=0} = \sum_\ell g''_\ell(q_\ell)[\delta q_\ell]^2$$

The above formulas prove (2.21) and (2.22). □

In carring out our estimates on the functional f it is useful to make use of the Riemann structure $\langle , \rangle$ on M which, as we have seen in Section 1, induces a infinite dimensional Riemann structure $\langle \cdot, \cdot \rangle_1$ on $\Lambda^1 M$.

Strictly related to $\langle\cdot,\cdot\rangle_1$, there is the functional (called energy functional)

$$E(\gamma) = \frac{1}{2}\int_{S^1} \langle\dot\gamma,\dot\gamma\rangle dt \tag{2.23}$$

Using a A-local representation, (2.23) takes the form

$$E(\gamma) = \frac{1}{2}\sum_{\ell=1}^{p}\int_{\tau_\ell} \sum_{ij} g_{ij}^{\ell}(q_\ell)\dot q_{\ell,i}\dot q_{\ell,j}\, dt \tag{2.24}$$

where q_ℓ, $\dot q_\ell$ is a A-local representation of $\dot\gamma$ and $\{g_{ij}^{\ell}\}$ is the metric tensor in the local coordinates of the chart $\{U_\ell,\phi_\ell\}$. $E(\gamma)$ is a particular case of the functional (2.1) when $L_t \circ \dot\gamma = \langle\dot\gamma,\dot\gamma\rangle$. So, by lemma 2.3 it follows that $E(\gamma)$ is a C^2-function of $\Lambda^1 M$.

<u>Lemma 2.4.</u> <u>There exist constants</u> a_1 <u>and</u> b_1 <u>such that</u>

$$\frac{1}{a_1} E(\gamma) - b_1 < f(\gamma) < a_1 E(\gamma) + b_1$$

<u>Proof</u>. Let L_ℓ be a local representation of L_t given by (0.3). For $\ell = 1,\dots,N$ we have

$$L_\ell(t,q,v) = L_\ell(t,q,v) + \sum_i \frac{\partial L_\ell}{\partial v_i}(t,q,u)v_i + \frac{1}{2}\sum_{ij}\frac{\partial L_\ell}{\partial v_i \partial v_j}(t,q,\theta\ v)v_i,v_j$$

where $\theta \in (0,1)$.

By the above formula, the compactness of M, and (L_3) we get

$$L_\ell(t,q,v) > -c_1 - c_2|v| + \frac{\nu}{2}|v|^2 > \frac{\nu}{4}|v|^2 - b_1$$

where c_1, c_2 and b_1 are suitable constants.

If g_{ij}^{ℓ} is the metric tensor of $\langle,\rangle$ in the chart U_ℓ, by the above inequality we get

$$L_\ell(t,q,v) > \frac{1}{a_1} g_{ij}^{\ell}(q)v_i v_j - c_3 \qquad \ell = 1,\dots,p\ .$$

where a_1 is a suitable constant.

The above inequality can be written as follows

$$L_t(\xi) > \frac{1}{a_1} \langle \xi,\xi \rangle - b_1 \quad \text{for every} \quad \xi \in TM$$

Taking $\gamma \in \Lambda^1 M$, $\xi = \dot{\gamma}$, integrating by the above inequality we get

$$f(\gamma) = \int_{S^1} L_t(\dot{\gamma}(t))dt > \frac{1}{a_1} \int_{S^1} \langle \dot{\gamma}, \dot{\gamma} \rangle \, dt - b_1 = \frac{1}{a_1} E(\gamma) - b_1$$

The other inequality can be obtained in an analogous way. □

The following lemma establishes estimates between intrinsic quantities and the corresponding quantities given by a A-local representation.

<u>Lemma 2.5</u>. <u>Let</u> γ, ξ, q, $\dot{q}$, δq, $\delta\dot{q}$ <u>as in Definition 2.1. Then there exists a constant</u> M <u>depending only on</u> A <u>and</u> $\langle \cdot,\cdot \rangle$ <u>such that</u>

$$\sum_{\ell=1}^{p} \|q\|^2_{W^1(\tau_\ell,R^n)} < M(1 + E(\gamma)) \tag{2.25}$$

$$\|\delta q\|_{L^\infty(\tau_\ell,R^n)} < M\|\xi\|^2_{\#} \quad \ell = 1,\ldots,p \tag{2.26}$$

$$\sum_{\ell=1}^{p} \|\delta q\|^2_{W^1(\tau_\ell,R^n)} > \frac{1}{M} \|\xi\|^2_1 - ME(\gamma)\|\xi\|^2_{\#} \tag{2.27}$$

<u>Proof</u>. By (2.3) we have $|q_\ell(t)| < c_1$ for every $t \in \tau_\ell$ ($\ell = 1,\ldots,p$). Then

$$\|q\|_{L^2(\tau_\ell,R^n)} < |t_\ell|c_1 \tag{2.28}$$

Since the atlas (0.2) is finite there is a constant c_2 such that

$$|\dot{q}_\ell(t)|^2 < c_2 g^\ell_{ij}(q_\ell(t))\dot{q}_{\ell,i}\dot{q}_{\ell,j} \quad \ell = 1,\ldots,p \; .$$

where g^ℓ_{ij} is the metric tensor.

Then we have

$$\sum_{\ell=1}^{p} \int_{\tau_\ell} |\dot{q}_\ell|^2 dt < c_2 \sum_{\ell=1}^{p} \int_{\tau_\ell} g^\ell_{ij}(q)\dot{q}_{\ell,i}\dot{q}_{\ell,j}dt = \tag{2.29}$$

$$= c_2 \sum_{\ell=1}^{p} \int_{\tau_\ell} \langle \dot{\gamma},\dot{\gamma} \rangle = c_2 E(\gamma)$$

By (2.28) and (2.29), (2.25) follows.

For $t \in \tau_\ell$ we have

$$\langle \xi(t), \xi(t) \rangle = \sum_{ij} g^\ell_{ij}(q)\,\delta q_{\ell,i}(t)\,\delta q_{\ell,j}(t)$$

then there is a constant c_3 such that

$$\frac{1}{c_3}\,|\delta q_\ell(t)|^2 \le \langle \xi(t), \xi(t) \rangle \le c_3 |\delta q_\ell(t)|^2$$

By the first of the above inequality (2.26) follows; by the second we get

$$\int \langle \xi(t), \xi(t) \rangle dt \le c_3 \sum_\ell \|\delta q\|^2_{L^2(\tau_\ell, R^n)} \tag{2.30}$$

For $t \in \tau_\ell$ we have

$$\langle \dot\xi, \dot\xi \rangle = \sum_{ij} g^\ell_{ij}(q)\,\nabla_t \delta q_{\ell,i}\,\nabla_t \delta q_{\ell,j} \tag{2.31}$$

where ∇_t denotes the covariant derivative:

$$\nabla_t \delta q_{\ell,i} = \delta \dot q_{\ell,i} + \sum_{h,k} \Gamma^i_{\ell,hk}(q_\ell)\,\dot q_{\ell,h}\,\delta q_{\ell,k} \tag{2.32}$$

where $\Gamma^i_{\ell,hk}$ are the Christoffel symbols relative to the chart U_ℓ. Then by (2.31) and (2.32) we get

$$\langle \dot\xi, \dot\xi \rangle \ge c_4 |\delta \dot q_\ell|^2 - c_5 |\dot q_\ell|\,|\delta q_\ell|$$

So integrating we get

$$E(\xi)(\gamma) = \int \langle \dot\xi, \dot\xi \rangle dt \ge c_4 \sum_\ell \|\delta q\|^2_{W^1(\tau_\ell, R^n)} - c_5 \sum_\ell \|\delta q_\ell\|_{L^\infty(\tau_\ell, R^n)} \|q\|_{W^1(\tau_\ell, R^n)}$$

Using (2.25) and (2.26), (2.27) follows. □

<u>Lemma 2.6</u>. <u>There are constants</u> a_2, b_2 <u>such that</u>

$$f''(\gamma)[\xi]^2 \ge \frac{1}{a_2}\|\xi\|_1^2 - b_2(1 + E(\gamma))\|\xi\|_\#^2$$

<u>Proof</u>. Using (2.22) and (2.15) we get

$$f''(\gamma)[\xi]^2 \ge \sum_{\ell=1}^{p} \left\{ a\|\delta q_\ell\|_{W^1(\tau_\ell, R^n)} - b(|\tau_\ell| + \|q_\ell\|^2_{W^1(\tau_\ell, R^n)})\|\delta q_\ell\|^2_{L^\infty(\tau_\ell, R^n)} \right\}$$

Then by (2.25) and (2.26) we get

$$f''(\gamma)[\xi]^2 > \frac{a}{M}\|\xi\|_1^2 - ME(\gamma)\|\xi\|_0^2 - bM\|\xi\|_0^2 \sum_\ell \left(|\tau_\ell| + \|q_\ell\|_{W^1(\tau_\ell,\mathbb{R}^n)}\right)$$

$$> \frac{a}{M}\|\xi\|_1^2 - ME(\gamma)\|\xi\|_0^2 + bM\|\xi\|_0^2(1 + M(1 + E(\gamma)))$$

$$= \frac{a}{M}\|\xi\|_1^2 - (M + bM^2)E(\gamma)\|\xi\|_0^2 + (bM + bM^2)\|\xi\|_0^2$$

The conclusion follows with $a_2 = \frac{M}{d}$ and $b_2 = \max(b,1)(M + M^2)$. □

<u>Lemma 2.7</u>. <u>Let</u> $\beta : [0,1] \to \Lambda^1 M$ <u>be a curve of class</u> C^1. <u>Then</u>

(a) $\frac{d}{d\lambda} E(\beta(\lambda)) < 2E(\beta(\lambda))^{1/2}\|\mathring{\beta}(\lambda)\|_1$

(b) $\frac{d}{d\lambda} E(\beta(\lambda))^{1/2} < \|\mathring{\beta}(\lambda)\|_1$

(c) $\int_0^1 E(\beta(\lambda))^{1/2} d\lambda < d_\beta$ where $d_\beta = \int_0^1 \|\mathring{\beta}(\lambda)\|_1 d\lambda$

(d) $\sqrt{E(\beta(0))} - \sqrt{E(\beta(1))} < \mathrm{dist}_1(\beta(0),\beta(1)) < d_\beta$

(e) <u>if</u> $\{\gamma_n\}$ <u>is a sequence such that</u> $E(\gamma_n)$ <u>is bounded, then there is a subsequence</u> γ_n' <u>converging in</u> ΛM.

<u>Proof</u>. (a) Define $\delta : [0,1] \times S^1 \to \Lambda^1 M$ as follows

$$\delta(\lambda,t) = [\beta(\lambda)](t)$$

Then we have

$$\frac{d}{d\lambda} E(\beta(\lambda)) = \frac{1}{2}\frac{d}{d\lambda}\int_0^1 \langle\partial_t\delta,\partial_t\delta\rangle dt$$

$$= \int_0^1 \langle\nabla_\lambda\partial_t\delta,\partial_t\delta\rangle dt \quad (\nabla_\lambda \text{ denotes the covariant derivative})$$

$$< \left(\int_0^1 \langle\nabla_\lambda\partial_t\delta,\nabla_\lambda\partial_t\delta\rangle dt\right)^{1/2}\cdot\left(\int_0^1 \langle\partial_t\delta,\partial_t\delta\rangle dt\right)^{1/2} \quad \text{(by the Schwartz inequality)}$$

$$< 2\|\mathring{\beta}(\lambda)\|_1\cdot E(\beta(\lambda))^{1/2}$$

(b) follows directly by (a).

(c) follows integrating (b)

(d) follows by (b) and the definition of $\mathrm{dist}_1(\cdot,\cdot)$

(e) by (d) we get that the sequence $\{\gamma_n\}$ is bounded in the metric $\langle\cdot,\cdot\rangle_1$. The conclusion follows by lemma 1.1. □

Let γ_0 and $\gamma_1 \in \Lambda^1 M$ two curves such that

(2.33) $$d_\#(\gamma_0,\gamma_1) < \rho$$

where ρ is small enough in order that the Riemann sphere $S_\rho(x)$ is geodesically convex for every $x \in M$. By virtue of the compactness of M and a well known theorem of J. Whitehead such ρ exists. Let $\delta : [0,1] \times S^1 \to M$ be a function such that

(a) $\delta(0,t) = \gamma_0(t)$; $\delta(1,t) = \gamma_1(t)$

(2.34) (b) $\lambda \to \delta(\lambda,t)$ is the shortest geodesic joining $\gamma_0(t)$ and $\gamma_1(t)$ parametrized with the arc length

By our assumption on ρ, δ is well defined. The fu tion δ defines a C^1-curve $\beta : [0,1] \to \Lambda^1 M$ in a natural way

(2.35) $$(\beta(\lambda))(t) = \delta(\lambda,t)$$

<u>Lemma 2.8</u>. <u>Let</u> β <u>be the curve defined by (2.35). Then</u>

$$\|\dot\beta(\lambda)\|_1 < (1 + a_0 d_\#^2)d_\beta \quad \text{for every} \quad \lambda \in [0,1]$$

<u>where</u> $\dot\beta(\lambda) = \frac{d}{d\lambda}\beta(\lambda)$, $d_\# = \mathrm{dist}_\#(\gamma_0,\gamma_1)$, $d_\beta = \int_0^1 \|\dot\beta(\lambda)\|_1 d\lambda$ <u>and</u> a_0 <u>is a constant which depends only on the Riemann manifold</u> $(M,\langle\cdot,\cdot\rangle)$.

<u>Remark</u>. In a linear space, where the tangent space can be identified with the space itself we have $\beta(\lambda) = (1-\lambda)\gamma_0 + \lambda\gamma_1$. Then $\|\dot\beta(\lambda)\|_1 = \|\gamma_1 - \gamma_0\|_1 = d_\beta$. Lemma 2.8 says that $\|\dot\beta(\lambda)\|_1$, in our situation, is not equal to d_β, but it can be nicely estimated.

<u>Proof</u>. By (2.34)(b) it follows that

(2.36) $$\nabla_\lambda \partial_\lambda \delta(\lambda,t) = 0 \quad \text{for every} \quad t \in S^1$$

(2.37) $$\langle\partial_\lambda\delta,\partial_\lambda\delta\rangle = \mathrm{dist}(\gamma_0(t),\gamma_1(t))^2 < d_\#^2 \quad \text{for every} \quad t \in S^1$$

We have

$$(2.38)\qquad \frac{d}{d\lambda}\|\dot\beta(\lambda)\|_1 = \frac{1/2}{\|\dot\beta(\lambda)\|_1}\frac{d}{d\lambda}\|\dot\beta(\lambda)\|_1^2 =$$

$$= \frac{1/2}{\|\dot\beta(\lambda)\|_1}\frac{d}{d\lambda}\int_0^1 \{\langle\nabla_t\partial_\lambda\delta,\nabla_t\partial_\lambda\delta\rangle + \langle\partial_\lambda\delta,\partial_\lambda\delta\rangle\}dt =$$

$$= \frac{1}{\|\dot\beta(\lambda)\|_1}\int_0^1 \{\langle\nabla_\lambda\nabla_t\partial_\lambda\delta,\nabla_\lambda\partial_t\delta\rangle + \langle\nabla_\lambda\partial_\lambda\delta,\partial_\lambda\delta\rangle\}dt =$$

$$= \frac{1}{\|\dot\beta(\lambda)\|_1}\int_0^1 \langle\nabla_\lambda\nabla_t\partial_\lambda\delta,\nabla_\lambda\partial_t\delta\rangle dt \qquad \text{by (2.36)}$$

By a well known formula of Riemannian geometry, if v is any vector field along δ, we have

$$(2.39)\qquad \nabla_\lambda\nabla_t v = \nabla_t\nabla_\lambda v - R_\delta(\partial_t\delta,\partial_\lambda\delta)v$$

where R is the Riemann curvature tensor. Moreover since our manifold M is compact, there exists a constant a_0 such that

$$(2.40)\qquad \langle R(v_1,v_2)v_3,v_4\rangle < a_0\|v_1\|\cdot\|v_2\|\cdot\|v_3\|\cdot\|v_4\|$$

where $v_i \in TM$ and $\|v_i\| = \langle v_i,v_i\rangle$. By (2.38), applying (2.39) with $v = \partial_\lambda\delta$ we get

$$(2.41)\qquad \left|\frac{d}{d\lambda}\|\dot\beta(\lambda)\|_1\right| < \frac{1}{\|\dot\beta(\lambda)\|_1}\left|\int_0^1 \{\langle\nabla_t\nabla_\lambda\partial_\lambda\delta,\nabla_t\partial_\lambda\delta\rangle - \langle R(\partial_t\delta,\partial_\lambda\delta)\partial_\lambda\delta,\nabla_\lambda\partial_t\delta\rangle\}dt\right|$$

$$< \frac{a_0}{\|\dot\beta(\lambda)\|_1}\int \|\partial_t\delta\|\cdot\|\partial_\lambda\delta\|^2\cdot\|\nabla_\lambda\partial_t\delta\|dt \quad \text{(by (2.36) and (2.40))}$$

$$< \frac{a_0 d_\#^2}{\|\dot\beta(\lambda)\|_1}\left(\int \|\partial_t\delta\|^2 dt\right)^{1/2}\cdot\left(\int \|\nabla_\lambda\partial_t\delta\|^2 dt\right)^{1/2}$$

(by (2.37) and the Schwartz inequality)

$$< a_0 d_\#^2 E(\beta(\lambda))^{1/2} \quad \text{(by the definition of } \|\dot\beta(\lambda)\|_1 \text{ and } E(\beta))$$

By the above formula we get

$$\|\mathring{\beta}(\bar{\lambda})\|_1 - \|\mathring{\beta}(\mu)\|_1 \le \left|\int_{\mu}^{\bar{\lambda}} \left|\frac{d}{d\lambda} \|\mathring{\beta}(\lambda)\|_1\right| dt\right| \le$$

$$\le a_0 d_\phi^2 \int_0^1 E(\beta(\lambda))^{1/2} d\lambda \le a_0 d_\phi^2 d_\beta \quad \text{(by Lemma 2.7(c).)}$$

Then, integrating the above formula in $d\mu$ we get

$$\|\mathring{\beta}(\bar{\lambda})\|_1 - d_\beta \le a_0 d_\phi^2 d_\beta$$

which proves the lemma. □

3. THE TOPOLOGY OF $\Lambda^1 M$

The topology of $\Lambda^1 M$ is strictly related to the topology of ΛM; in fact we have the following theorem

Theorem 3.1. The embedding

$$i : \Lambda^1 M \to \Lambda M$$

is a homotopy equivalence.

Proof. See [K] Th. 1.2.10. □

For our purposes, by virtue of Theorem 3.1 it is enough to study the topology of ΛM. We have the following results of Vigue-Poirrier and Sullivan:

Theorem 3.2. If $\pi_1(M) = 0$ there exists an infinite set of positive integer

$$Q \subset \mathbb{N}$$

such that

$$H^q(\Lambda M) \neq 0 \quad \text{for every} \quad q \in Q$$

where $H^q(\Lambda M)$ is the cohomology ring with real coefficients.

Proof. If the cohomolgy algebra $H^*(M)$ requires at least two generators, then the result follows from the main theorem of [V.P.S.] on page 637.

If $H^*(M)$ has only one generator, the result follows from the Addendum of [V.P.S] on page 643. □

By the above theorem and theorem 3.1, the following corollary follows

Corollary 3.3. Under the same assumptions of theorem 3.2

$$H_q(\Lambda^1 M) \neq 0 \quad \text{for every} \quad q \in Q$$

Now let $\rho > 0$ be small enough in order that the Riemann sphere $S_\rho(x)$ is geodesically convex for every $x \in M$. We set

$$E_c = \{\gamma \in \Lambda^1 M | E(\gamma) < c\} \tag{3.1}$$

The following result holds.

Theorem 3.4. E_c is homotopically equivalent to a manifold N of dimension less or equal to $(\dim M)(\frac{\sqrt{\rho}}{c} + 1)$.

Proof. The proof is essentially the same of the proof of Theorem 16.2 of Milnor [M]. Actually instead of using the manifold $\Lambda^1 M$, he uses the (non-complete) manifold of broken geodesics, but its proof can be adapted to our situation without major changes. We shall give a sketch of it. Let $S_\rho(x)$ be the Riemann ball of radius ρ and center x. By virtue of the compactness of M and well known theorems, it is possible to choose ρ small enough in order that $S_\rho(x)$ is geodesically convex for every $x \in M$. We now set

$$\tilde{E}_c = \{\gamma \in E_c | \gamma|_{[t_{i-1},t_i]} \text{ is a geodesic for } i = 1,\dots,N\}$$

where $t_i = \frac{i}{N}$ and N satisfies $\frac{\sqrt{c}}{\rho} < N < \frac{\sqrt{c}}{\rho} + 1$. Notice that, by virtue of our restriction on N, if $\gamma \in \tilde{E}_c$, $\gamma([t_{i-1},t_i])$ is contained in $S_\rho(x)$ for some $x \in M$. Now we want to show that $\tilde{E}_c$ is a finite dimensional manifold. To do this we set

$$\Delta = \{(x_1,\dots,x_N) \in M^N | \mathrm{dist}(x_{i-1},x_i) < \rho \quad i = 1,\dots,N\}$$

and consider the map

$$\pi : \Delta \to \tilde{E}_c$$

defined as follows

$$\pi(x_1,\dots,x_N) = \gamma \quad \text{with} \quad \gamma(t_i) = x_i$$

This map is obviously continuous since x_{i-1} and x_i belong to $S_\rho(x)$ for some $x \in M$ and since $S_\rho(x)$ is geodesically convex, the (unique) geodesic which join x_{i-1} and x_i depends continuously on x_i and x_{i+1}. Moreover it is invertible, in fact

$$\pi^{-1}(\gamma) = (\gamma(t_1),\dots,\gamma(t_N)) .$$

This proves that $\tilde{E}_c$ is a manifold of dimension $(\dim M)\cdot\left(\left[\frac{\sqrt{c}}{\rho}\right] + 1\right)$ where $[a]$ denotes the integer part of a. The next step will be to prove the $\tilde{E}_c$ is a deformation retract of E_c. The retraction $r : \left[0, \frac{1}{N}\right] \times E_c \to \tilde{E}_c$ is defined as follows

$$r(\lambda,\gamma)(t) = \begin{cases} \text{the unique geodesic joining } \gamma(t_i) \text{ with } \gamma(t_i + \lambda) & \text{for } t \in [t_i,t_i + \lambda] \\ \\ \gamma(t) \quad \text{for } t \in [t_i + \lambda, t_{i+1}] & i = 0,\dots,N - 1 . \end{cases}$$

If you remember that $t_i = \frac{i}{N}$, the above definition makes sense for $\lambda \in \left[0, \frac{1}{N}\right]$. Clearly $r(0,\gamma) = \gamma$ and $r\left(\frac{1}{N}, t\right) \in \tilde{E}_c$. Moreover, it is easy to see that r is continuous in $\left[0, \frac{1}{N}\right] \times \Lambda^1 M$ and it is equal to the identity for $\gamma \in \tilde{E}_c$. This proves the theorem. □

By Theorem 3.4 the following conclusion follows straightforward.

<u>Corollary 3.5</u>. $H^k(E_c) = 0$ *for* $k > (\dim M)\left[\frac{\sqrt{c}}{\rho} + 1\right]$.

4. THE MAIN RESULTS.

We recall the well known assumption (c) of Palais and Smale (which will call P.S.)

Definition 4.1. Let X be a Riemann manifold modelled on an Hilbert space and let $f \in C^1(X,\mathbb{R})$. We say that $\{X,f\}$ satisfies P.S. if any sequence $\gamma_n \in X$ such that $f(\gamma_n) \to c$ and $\nabla f(\gamma_n) \to 0$ has a converging subsequence.

The above condition is used to prove the following well known theorem:

Theorem 4.2. Let $\{X,f\}$ satisfy P.S. and let Γ be a family of subsets of X such that

(a) $A \in \Gamma$ such that $f|_A$ is bounded from above.

(b) $\forall A \in \Gamma$ $f|_A >$ const.

(c) if η is a deformation of X, (i.e. it is a homeomorphism on X homotopic to the identity) then $A \in \Gamma$ if and only if $\eta(A) \in \Gamma$.

Under such assumption

$$c = \inf_{A \in M} \sup_{\gamma \in A} f(\gamma)$$

is well defined and it is a critical value of f.

Our goal is to apply theorem 4.2 to the couple $\{\Lambda^1 M, f\}$ where f is defined by (2.1). The first step is to prove the following lemma.

Lemma 4.3. $\{\Lambda^1 M, f\}$ satisfies P.S.

Proof. First of all we remark that ∇f, given by the formula

$$\langle \nabla f(\gamma), \xi \rangle_1 = f'(\gamma)[\xi]$$

is well defined and continuous by lemma 2.3. Now let $\{\gamma_n\}$ be a sequence such that

$$(4.1) \qquad \begin{aligned} &(a) \quad f(\gamma_n) \to c \\ &(b) \quad \nabla f(\gamma_n) \to 0 \end{aligned}$$

By (4.1)(a) and lemma 2.4, it follows that $E(\gamma_n)$ is bounded. So by lemma 2.7(e), we can consider a subsequence which is a Cauchy sequence in ΛM. We shall denote this subsequence again with γ_n. We want to show that $\{\gamma_n\}$ is a Cauchy sequence in $\Lambda^1 M$. We chose $\varepsilon > 0$ and N large enough in order that, for $m,n > N$ we have

$$(4.2)\qquad \begin{aligned}&(a)\quad \|\nabla f(\gamma_n)\| < \frac{\sqrt{\varepsilon}}{4a_2}\\ &(b)\quad d_{\#}(\gamma_n,\gamma_m) < \min\left(\rho, \sqrt{\frac{1}{a_0}}, \sqrt{\frac{\varepsilon}{a_2b_2(E+1)}}, \frac{1}{4}\sqrt{\frac{1}{a_2b_2}}\right)\end{aligned}$$

where $E = \sup_{n\in\mathbb{N}} E(\gamma_n)$; ρ, a_0, a_2 and b_2 are the constants appearing in (2.33) and lemmas 2.8 and 2.6. Now let $\beta : [0,1] \to \Lambda^1 M$ be a curve defined by (2.35) and (2.34 with $\beta(0) = \gamma_n$ and $\beta(1) = \gamma_m$. Moreover set, as in lemma 2.8 $d_\beta = \int_0^1 \|\dot\beta(\lambda)\|_1 d\lambda$ and $d_{\#} = \mathrm{dist}_{\#}(\gamma_n,\gamma_m)$. Clearly we have

$$\|\dot\beta(\lambda)\|_{\#} = d_{\#}$$

and by lemma 2.6 we have

$$(4.2')\qquad f''(\beta(\lambda))[\dot\beta(\lambda)]^2 > \frac{1}{a_2}\|\dot\beta(\lambda)\|_1^2 - b_2 d_{\#}^2(1 + E(\beta(\lambda))) .$$

So we have

$$\begin{aligned}(4.3)\qquad d_\beta^2 &< \int_0^1 \|\dot\beta(\lambda)\|_1^2 d\lambda \quad \text{(by Schwartz inequality)}\\ &< \int_0^1 \left\{a_2 \frac{d^2}{d\lambda^2} f(\beta(\lambda)) + a_2b_2d_{\#}^2(1 + E(\beta(\lambda)))\right\}d\lambda \quad \text{(by (4.2'))}\\ &< a_2(|\langle\nabla f(\gamma_m),\dot\beta(0)\rangle| + |\langle\nabla f(\gamma_m),\dot\beta(1)\rangle|) + a_2b_2d_{\#}^2 + a_2b_2d_{\#}^2 \int_0^1 E(\beta(\lambda))d\lambda\\ &\qquad\qquad\text{(by an integration in } \lambda)\end{aligned}$$

Also we have

$$\begin{aligned}(4.4)\qquad & a_2(|\langle\nabla f(\gamma_m),\dot\beta(0)\rangle| + |\langle\nabla f(\gamma_m),\dot\beta(1)\rangle|) <\\ &< a_2(\|\nabla f'(\gamma_n)\|\cdot\|\dot\beta(0)\| + \|\nabla f(\gamma_m)\|\cdot\|\dot\beta(1)\|) <\\ &< 2a_2 \cdot \frac{\sqrt{\varepsilon}}{4a_2} \cdot (1 + a_0 d_{\#}^2) d_\beta \quad \text{(by (4.2)(a) and lemma 2.8)}\\ &< \sqrt{\varepsilon}\cdot d_\beta \quad \text{(by (4.2)(b))}\\ &< \frac{1}{4} d_\beta^2 + \varepsilon\end{aligned}$$

Also we have

$$(4.5)\qquad E(\beta(\lambda)) = E(\gamma_n) + \int_0^\lambda \frac{d}{d\lambda} E(\beta(\tau))d\tau$$

$$\leq E + 2\int_0^1 E(\beta(\tau))^{1/2}\|\dot\beta(\tau)\|d\tau \quad \text{(by lemma 2.7(a) and the definition of } E)$$

$$\leq E + 2(1 + a_0 d_\#^2)d_\beta \cdot \int_0^1 E(\beta(\tau))^{1/2}d\tau \quad \text{(by lemma 2.8)}$$

$$\leq E + 2(1 + a_0 d_\#^2)d_\beta^2 \quad \text{(by lemma 2.7(c))}$$

$$\leq E + 4d_\beta^2 \quad \text{(by 4.2(b))}$$

So by (4.3), (4.4), (4.5) and (4.2) we get

$$d_\beta^2 \leq \frac{1}{4}d_\beta^2 + \varepsilon + a_2 b_2 d_\#^2 + a_2 b_2 d_\#^2 E + 4a_2 b_2 d_\#^2 b_\beta^2 \leq \frac{1}{4}d_\beta^2 + 3\varepsilon + \frac{1}{4}d_\beta^2 = \frac{1}{2}d_\beta^2 + 3\varepsilon$$

Thus

$$d_\beta^2 \leq 6\varepsilon$$

Since $d_\beta \geq \mathrm{dist}_1(\gamma_n,\gamma_m)$, by the arbitrariness of ε the conclusion follows. □

For any set $A \subset \Lambda^1 M$ let $i_A : A \to \Lambda^1 M$ denote the natural embedding and let $i^*_{k,A} : H^k(\Lambda^1 M) \to H^k(A)$ induced homomorphism. Let Q be the set defined in Theorem 3.2. Then for every $k \in Q$ we set

$$(4.6)\qquad \Gamma^k = \{A \in \Lambda^1 M \mid i^*_{k,A} \neq 0\}$$

<u>Theorem 4.3</u> <u>If</u> $\pi_1(M) = 0$, <u>for every</u> $k \in Q$, <u>the number</u>

$$c_k = \inf_{A\in\Gamma^k} \sup_{\gamma\in A} f(\gamma)$$

<u>is well defined and it is a critical value of</u> f. <u>Moreover,</u>

$$(4.7)\qquad \lim_{\substack{k\in Q\\ k\to+\infty}} c_k = +\infty$$

<u>Proof</u>. In order to prove the first part of the theorem, it is sufficient to apply Theorem 4.2 with $X = \Lambda^1 M$. $\{\Lambda^1 M, f\}$ satisfies P.S. by lemma 4.3. By corollary 3.3 it follows that the sets $\Gamma^k (k \in N)$ are not empty and contain compact sets (in fact they contain the support of k-chains which are not homologous to a constant). Then the assumption (a) of

Theorem 4.2 is satisfied. By virtue of lemma 2.4, f is bounded from below on $\Lambda^1 M$. Then assumption (b) follows. Assumption (c) follows from the fact that η induces a isomorphism η^* which makes the following diagram to commute:

$$\begin{tikzcd}
 & H^k(\Lambda^1 M) \arrow[dl, "i^*_{k,A}"'] \arrow[dr, "i^*_{k,\eta(A)}"] & \\
H^k(A) & & H^k(\eta(A)) \arrow[ll, "\eta^*", "\cong"']
\end{tikzcd}$$

So $i^*_{k,\eta(A)} = (\eta^*)^{-1} \circ i^*_{k,A} \neq 0$ if and only if $i^*_{k,A} \neq 0$. So, by Theorem 4.2, the first part of Theorem 4.3 follows. In order to prove (4.7), we fix $k \in Q$, $\varepsilon > 0$ and we take $\bar{A} \in \Gamma^k$ such that

$$\sup_{\gamma \in \bar{A}} f(\gamma) \le c_k + \varepsilon$$

For $\gamma \in \bar{A}$, by lemma 2.4, it follows that

$$E(\gamma) \le a_2 f(\gamma) + b_2 = a_2(c_k + \varepsilon) + b_2$$

So, setting $c = a_2(c_k + \varepsilon) + b_2$, we have that $\bar{A} \xrightarrow{i_1} E_c$ where E_c is defined by (3.1). Then we obtain the following commutative diagram

$$\begin{tikzcd}
H^k(\Lambda^1 M) \arrow[rr, "i^*_{\bar{A}}"] \arrow[dr, "i^*_2"'] & & H^k(\bar{A}) \\
 & H^k(E_c) \arrow[ur, "i^*_1"'] &
\end{tikzcd}$$

where $i_2 : E_c \to \Lambda^1 M$ is the embedding. Since $\bar{A} \in \Gamma^k$, $i_{\bar{A},k} \neq 0$; then $i^*_1 \neq 0$. Therefore $H^k(E_c) \neq 0$. Then by Corollary 3.5 it follows that

$$k \le \dim M\left(\frac{\sqrt{c}}{\rho} + 1\right)$$

Then by the definition of c, we obtain that

$$c_k > \frac{k^2}{(\dim M)^2}\rho^2 - M \quad (M \text{ is a positive constant}).$$

This proves (4.7). $\square$

Proof of Theorem 0.1. (a) A connected component of ΛM corresponds to every conjugacy class α of $\pi_1(M)$ and by virtue of Theorem 3.1, a connected component $C(\alpha)$ of $\Lambda^1 M$. Define

$$c_\alpha = \inf_{\gamma \in C(\alpha)} f(\gamma)$$

Since $\{\Lambda^1 M, f\}$ satisfy P.S., then c_α is a minimum and, of course, it is a critical value of f. Moreover, if $\alpha \neq \alpha'$, the critical points of f are distinct since they belong to different connected components.

(b) If $\pi_1(M) = 0$, then the conclusion follows by Theorem 4.3. Otherwise consider the universal covering space $\tilde{M} \xrightarrow{\pi} M$. Since $\pi_1 M$ is finite, $\tilde{M}$ is compact. Let $\tilde{L}(t) = L(t) \circ T\pi$ for every $t \in [0,1]$. Then $\tilde{M}$ and $\tilde{L}(t)$ satisfy the assumptions of Theorem 4.3. Therefore there are infinitely many periodic orbit $\tilde{\gamma}_k$ of $\tilde{L}(t)$. Clearly $\pi\tilde{\gamma}_k$ is a periodic orbit of $L(t)$, and by its construction it is homotopically trivial. $\square$

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Let M be a smooth n-dimensional manifold and let TM be its tangent bundle. We consider a time periodic Lagrangian of period T,

$$L_t : TM \to R$$

(continued)

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ABSTRACT (cont.)

and we seek T-periodic solutions of the Lagrange equations, which in local coordinates are

$$(*) \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(t,q,\dot{q}) - \frac{\partial L}{\partial q}(t,q,\dot{q}) = 0 \qquad i = 1,\dots,n \ .$$

Our main result states that if the fundamental group of M is finite, then (*) has infinitely many T-periodic solutions, provided that L_t satisfies certain physically reasonable assumptions.